Instantaneous Information Propagation in a Traffic Stream through Inter-Vehicle Communication

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Monte Carlo Simulation of Instantaneous Information Propagation through Inter-Vehicle Communication in a Traffic Stream

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Abstract

De-centralized traffic information systems based on inter-vehicle communications have drawn more and more attention from researchers and operators in the areas of transportation engineering and information technology with the development of wireless communication technologies. A fundamental performance measurement of such systems is the probability for a piece of information to travel beyond a point. In (Jin and Recker, 2005), based on the observation and assumption of instantaneous information propagation in a traffic stream, a novel analytical model was proposed to analyze the lower bound of the success rate for information to travel beyond a point under certain traffic conditions, penetration rate, and communication range. In this paper, we propose another model of the lower bound by using Monte Carlo simulation. With three different, well-chosen random number generators, we demonstrate that Monte

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Carlo simulation results converge to those of the analytical model with increasing number of Monte Carlo experiments, in a manner predicted by the fundamental theorem of Monte Carlo simulation. In this sense, the two models cross-validate each other. Hence, both the Monte Carlo simulation model and the analytical model can be used in practice to estimate the performance of an inter-vehicle communication system or select appropriate communication devices and technologies to realize a level of system performance.

1 Introduction

In recent years, ad hoc communication networks formed by mobile nodes have been investigated extensively (Perkins, 2000). Interesting applications of such technologies include distributed sensor networks and inter-vehicle communication (IVC) systems. Especially, IVC systems have been envisioned to be the basis of the next generation traffic information systems, which are decentralized in nature and more resilient to disasters such as earthquakes. Different from other mobile ad hoc networks, an IVC system consists of highly correlated communication nodes, i.e. equipped vehicles, since movements of vehicles are governed by rules such as car-following (Gazis et al., 1961) and traffic flow patterns such as shock waves are common on a road (Lighthill and Whitham, 1955).

To better understand the benefits of an IVC system to traffic control and management, researchers at the California Institute of Telecommunications and Information Technology and the Institute of Transportation Studies of the University of California, Irvine, have been engaged in comprehensive research efforts aimed at the development of an autonomous, self-organizing, transportation management, information, and control system (Autonet). The major goals are for studying the influence of IVC on drivers' behaviors in a road network, such as route choice and departure time choice behaviors. In these studies, one thrust of research has been to identify and measure the most important features of an Autonet system. For example, how far a piece of information can travel in such a system under certain traffic conditions? To answer this question, one approach is to repeat experiments by incorporating traffic simulators (Hartenstein et al., 2001; Yang, 2003). In another approach developed in (Jin and Recker, 2005), based on the observation that a piece of message of reasonable size travels much faster than vehicles, information propagation was assumed to be instantaneous relative to traffic dynamics. Then a novel model was proposed to analytically describe the relationship between the so-called success rate, a measurement of connectivity between vehicles, and traffic conditions, penetration rate, and communication range.

In the model in (Jin and Recker, 2005), parameters include traffic conditions (measured by traffic density ρ), transmission range R, and penetration rate μ. To measure the best possible performance of inter-vehicle communication, information is considered to propagate in the manner of "most forwarded within range" (MFR) (Takagi and Kleinrock, 1984). For a traffic stream with a number of equipped vehicles, one can obtain a so-called MFR communication chain. After studying the properties of all possible MFR communication chains in a traffic stream for different locations of equipped vehicles, we proposed a regressive model to compute the success rate for information to travel h hops to vehicle (c, k), denoted by $P(c, k; h)$. Further we defined the probability for information to travel beyond (c, k) at h hops as

$$S(c, k; h) = \sum_{(d,i)} P(d, i; h),$$

which can be considered as relative success rate, and the lower bound of the absolute success rate for information to travel beyond (c, k) by

$$s(c, k) = \max_{h} S(c, k; h).$$

The lower bound was therefore used as a measurement of the performance of an inter-vehicle communication communication system. In (Jin and Recker, 2005), the success rate $s(c, k)$ is studied for uniform traffic, randomly distributed traffic, and traffic streams with gaps and shock waves. The study showed that the lower bound of absolute success rate for information to reach beyond a point is consistent in magnitude with the corresponding results in (Hartenstein et al., 2001; Yang, 2003).

This consistency suggests that the observation and assumption of instantaneous information propagation is valid. However, since the measurement of performance and concepts and the approach in the model of (Jin and Recker, 2005) are significantly different from those in literature, the novel analytical model is subject to careful validation.

In this paper, we propose another model of instantaneous IVC by Monte Carlo simulation (Gentle, 2003). Monte Carlo simulation has been widely applied in studying characteristics of complicated processes, and, if properly devised, should provide more accurate results with larger number of samples. In an IVC system, where equipped vehicles are randomly located, it is natural to use Monte Carlo simulations to study its properties, such as how far a piece of information can travel. Studies in (Hartenstein et al., 2001; Yang, 2003) can both be considered Monte Carlo study in a loose sense. However, there are the following differences in the model developed in this paper with those in literature. First, in literature, traffic dynamics models are used. Since randomness can be introduced in traffic simulators, simulation results are not only related to the random distribution of equipped vehicles but also to randomness in traffic dynamics. In our study, in contrast, we assume no traffic dynamics relative to information propagation, and only randomness in distribution of equipped vehicles is considered. Therefore, the models in literature are fundamentally different from the one developed in this study. Second, convergence of Monte Carlo simulations was not checked in existing studies, since the computational load in traffic simulators in those studies prevents a large number of simulations. In our study, we will carry out the convergence test with as many as 10^8 Monte Carlo experiments, which can be finished in a reasonable amount of time.

The rest of the paper is organized as follows. In Section 2, we introduce three well-chosen random number generators. In Section 3, we formulate the instantaneous information propagation problem with Monte Carlo simulation and propose measurements of simulation errors. In Section 4, with three random number generators, we demonstrate simulation results and compare simulation results with theoretical results. In Section 5, we conclude our study.

2 Three random number generators

To ensure the validity of results of Monte Carlo simulations, general requirements on used random number generators (RNGs) include long period, good distribution properties, and fast computing speed. Moreover, chosen RNGs should be thoroughly analyzed and tested theoretically and empirically. In addition, since every RNG has its own known or unknown limitations and could intrinsically interfere with a Monte Carlo simulation model, it is recommended to use several different RNGs in a study. [1]

In our study, we choose three RNGs: the standard RAND() function included in C-library stdlib.h, MRG32k3a (L'Ecuyer, 1999), and MT19937 (Matsumoto and Nishimura, 1998). The first is a simple C-function and can be easily implemented, while the latter two, with carefully chosen parameters, have been theoretically and empirically shown to have long periods, good distribution properties, and fast in computing. In the following, the three RNGs are briefly introduced.

2.1 RAND RNG

RAND is a standard RNG in the C-library stdlib.h and realized by a linear congruential method (Knuth, 1997) in the form of

$$x_{n+1} = (ax_n + b) \bmod m, \tag{1}$$

where a, b, and m are constants. On a 32-bit machine, the period of this RNG is about 2^{31} (Matsumoto and Nishimura, 1998).

2.2 MRG32k3a RNG

The MRG32k3a (L'Ecuyer, 1999) algorithm, a combined multiple recursive random number generator (MRG), combines three copies of multiple recursive random number generator in the following

[1] Ref: http://random.mat.sbg.ac.at/generators/index.html

form ($j = 1, 2$):

$$x_{j,n} = (a_{j,1}x_{j,n-1} + a_{j,2}x_{j,n-2} + a_{j,3}x_{j,n-3}) \mod m_j, \tag{2a}$$

$$z_n = (x_{1,n} - x_{2,n}) \mod m_1, \tag{2b}$$

$$\tilde{u}_n = (z_n + \delta m_1)/(m_1 + 1), \tag{2c}$$

$$\delta = \begin{cases} 0, & z_n \neq 0; \\ 1, & z_n = 0. \end{cases} \tag{2d}$$

In this RNG, the two moduli $m_1 = 2^{32} - 209$ and $m_2 = 2^{32} - 22853$ are distinct primes, the jth MRG has period length of $m_j^3 - 1$, $a_{1,1} = 0$, $a_{1,2} = 1403580$, $a_{1,3} = -810728$, $a_{2,1} = 527612$, $a_{2,2} = 0$, $a_{2,3} = -1370589$, and initial values (or seed) of $(x_{1,0}, x_{1,1}, x_{1,2})$ and $(x_{2,0}, x_{2,1}, x_{2,2})$ must not be all zero and must be smaller than both m_1 and m_2 respectively. The MRG32k3a RNG returns a uniformly distributed random number $\tilde{u}_n \in (0, 1)$, with period length of $(m_1^3 - 1)(m_2^3 - 1)/2 \approx 2^{191}$. Since $a_{j,i}(m_j - 1) < 2^{53}$ for $j = 1, 2$ and $i = 1, 2, 3$, quantities in (2a) can always be exactly represented by 32-bit floating point numbers on a computer supporting the IEEE 754 floating-point arithmetic standard, on which a number of double precision has 53 bits for the significand (Goldberg, 1991). Therefore, in the implementation of MRG32k3a, all variables are converted to exact floating-point representations (e.g., in C-language compiler), and all computations are directly done with floating-point arithmetic. This implementation is generally faster than implementation with integer arithmetic. Note that, however, even though $\tilde{u}_n$ is represented with 53-bit floating-point numbers, the actual output string is just 32 bits due to rounding errors, and, to obtain more precision, one can combine two or more strings together. In (L'Ecuyer, 1999), the MRG32k3a RNG has been shown to have good structural properties through spectral test (Knuth, 1997).

2.3 MT19937 RNG

The MT19937 RNG, Mersenne Twister 19937 (Matsumoto and Nishimura, 1998), is a type of multiple-recursive matrix method based on boolean arithmetics. In a recommended implementation, a sequence of 32-bit floating-point or integer numbers, $x_k = (x_k^{31}, \cdots, x_k^0)$, where x_k^i is a boolean number ($i = 0, \cdots, 31$), are generated as follows, where all computations are in boolean arithmetics, and $x << u$ and $x >> u$ mean u-bit shift left and right respectively

$$x_{k+624} = x_{k+397} + (x_k^{31}, x_{k+1}^{30}, \cdots, x_{k+1}^{0}) \cdot A, \tag{3a}$$

$$z_{k+624} = x_{k+624} \cdot T, \tag{3b}$$

where

$$x \cdot A = x >> 1 + x^0 \cdot 9908B0DF, \tag{3c}$$

and the tempering operator T is defined in order as

$$y = x + (x >> 11), \tag{3d}$$

$$y = y + (y << 7) \text{ AND } 9D2C5680, \tag{3e}$$

$$y = y + (y << 15) \text{ AND } EFC60000, \tag{3f}$$

$$z = y + (y >> 18). \tag{3g}$$

With the definition above, MT19937 has 623-dimensional equidistribution property and an exceedingly large period of $2^{19937} - 1$, where 19937 is the 24th known Mersenne number [2]. Due to computational difficulty of implementing spectral test for the 623-dimensional RNG, k-distribution test was used as the major statistical test. In MT19937, favorable k-distribution property is obtained through the tempering process in (3b), since the raw sequence obtained from sole linear recurrence in (3a) has very poor k-distribution property. In addition, MT19937 also passed other standard tests, such as the diehard tests [3]. With the aforementioned features, MT19937, consuming 624 computer word, was reported to be as fast as RAND since it is based on boolean arithmetics. Therefore, it is suggested that MT19937 could be most suitable for Monte Carlo simulations of complicated

[2] Ref: http://www.utm.edu/research/primes/mersenne/

[3] Ref: http://www.csit.fsu.edu/∼burkardt/f_src/diehard/diehard.html

systems (Matsumoto and Nishimura, 1998). In our study, we use an implementation of MT19937 in C, available at http://www.math.sci.hiroshima-u.ac.jp/∼m-mat/MT/MT2002/emt19937ar.html.

3 Monte Carlo simulation of instantaneous information propagation

For a traffic stream with n vehicles in C cells and vehicle (C, K) as the last vehicle, the flow-chart of Monte Carlo simulation of instantaneous information propagation is shown in Figure 1, in which each step is carried out as follows.

- Inputs for the experiments include positions of n vehicles $x(c, k)$, penetration rate μ, communication range R, and the number of experiments M.
- In each experiment, n uniformly distributed random variables, among which random variable $X_{(c,k)}$ corresponds to vehicle (c, k), are generated in $[0, 1]$.[4]
- If $X_{(c,k)} \leq \mu$, vehicle (c, k) is equipped, and the event when vehicle (c, k) is equipped is denoted by $(c, k; -1)$.
- Starting from the source, information is transmitted to the farthest equipped vehicle within a communication range, i.e., in the manner of MFR. In this way, we can obtain a unique MFR communication chain for each experiment. If information travels to vehicle (c, k) with h hops $(h > 0)$, or vehicle (c, k) is the hth node of a communication chain, we denote this event by $(c, k; h)$.
- The numbers of occurrences of $(c, k; -1)$ and $(c, k; h)$ are denoted by $N(c, k; -1)$ and $N(c, k; h)$ respectively. In each experiment, these quantities are added by 1 if the corresponding events occur.

[4]The interval can also be $(0, 1)$, $(0, 1]$, or $[0, 1]$.

- After finishing M experiments, the average probabilities of $(c,k;-1)$ and $(c,k;h)$ can be computed by

$$\begin{aligned} P_M(c,k;-1) &= \frac{N(c,k;-1)}{M}, && (4)\\ P_M(c,k;h) &= \frac{N(c,k;h)}{M}. && (5)\end{aligned}$$

Then we can further compute the following performance measurements of an IVC system

$$\begin{aligned} S_M(c,k;h) &= \sum_{(d,i)=(c,k)}^{(C,K)} P_M(c,k;h), && \text{(6a)}\\ s_M(c,k) &= \max_h S_M(c,k;h), && \text{(6b)}\end{aligned}$$

where $S_M(c,k;h)$ is the success rate relative to the number of hops, and $s_M(c,k)$ is the lower bound of the absolute success rate for information to travel beyond vehicle (c,k).

To compare the difference between $P_M(c,k;-1)$ and μ, for $(c,k)=(1,1),\cdots(C,K)$, we define the following three aggregate errors:

$$\begin{aligned} \|P_M(c,k;-1)-\mu\|_1 &= \frac{\sum_{(c,k)=(1,1)}^{(C,K)} |P_M(c,k;-1)-\mu|}{K},\\ \|P_M(c,k;-1)-\mu\|_2 &= \sqrt{\frac{\sum_{(c,k)=(1,1)}^{(C,K)} |P_M(c,k;-1)-\mu|^2}{K}},\\ \|P_M(c,k;-1)-\mu\|_\infty &= \max_{(c,k)=(1,1)}^{(C,K)} |P_M(c,k;-1)-\mu|.\end{aligned}$$

Similarly, we can define these errors between $P_M(c,k;h)$ and $P(c,k;h)$. Since $P(c,k;h)$ is a two-dimensional vector with $h=1,\cdots,2C-1$ and $(c,k)=(1,1),\cdots,(C,K)$, the summation in the equations above has to be taken for both the number of vehicles and the number of hops. Since the two performance measurements in (6) are derived from $P_M(c,k;h)$, we expect them to have similar convergence pattern as $P_M(c,k;h)$.

4 Monte Carlo simulation results

In this section, we simulate instantaneous information propagation with Monte Carlo experiments. We use the following seeds for the three RNGs: 12345 for initial value x_1 in RAND, 12345 for all initial values of $(x_{1,0}, x_{1,1}, x_{1,2})$ and $(x_{2,0}, x_{2,1}, x_{2,2})$ in MRG32k3a, and four initial values 0x123, 0x234, 0x345, 0x456 in MT19937 [5]. Since all seeds are constant, these simulation results can be properly replicated by any interested parties. Here, the number of experiments, M, varies from 10^3 to 10^8.

4.1 Monte Carlo simulation for a uniform traffic stream

In this subsection, we show Monte Carlo simulation results with penetration rate $\mu = 10\%$ and communication range $R = 1$ km for a uniform traffic stream, spreading 15 communication cells with 300 vehicles [6]. In this subsection, we only use MT19937 RNG to demonstrate the properties of the Monte Carlo simulation model qualitatively.

In Figure 2, we demonstrate the differences in $P(c, k; h)$ and $s(c, k)$ between Monte Carlo simulation results and theoretical results for hops $h = 2$, 6, and 10, with the number of Monte Carlo experiments $M = 10^3$, 10^4, 10^5, and 10^6. From the figure, we have the following qualitative observations. First, for the same number of experiments, $|P_M(c, k; h) - P(c, k; h)|$ decreases with the number of hops h. Since the absolute values of $P(c, k; h)$ and $P_M(c, k; h)$ also decrease with h, we expect the relative errors to be comparable for different number of hops. Second, Monte Carlo simulation results of $P_M(c, k; h)$ and $s_M(c, k)$ get closer to their analytical counterparts, as we increase the number of experiments. This suggests that Monte Carlo simulation results converge to the theoretical results. Therefore, qualitatively, the Monte Carlo simulation results and the theoretical results are consistent with each other.

[5]Ref: http://www.math.sci.hiroshima-u.ac.jp/∼m-mat/MT/MT2002/CODES/mt19937ar.c

[6]This traffic stream is the same as in Figure 3 of (Jin and Recker, 2005).

4.2 Monte Carlo simulation for a non-uniform traffic stream

In this subsection, we show Monte Carlo simulation results for a randomly generated traffic stream, the same as in Figure 6 of (Jin and Recker, 2005), which spreads 15 communication cells and consists of 1422 vehicles. We run Monte Carlo simulation on a 32-bit Windows-xp desktop with 3GHz Pentium-4 CPU and 1GB RAM. The Monte Carlo simulation model is implemented with the MINGW32 c/c++ compiler[7]. Here we carry out Monte Carlo simulations and compare the results with the theoretical model for the three RNGs discussed in the preceding section: RAND, MRG32k3a, and MT19937.

For the three RNGs, simulation results are shown in Tables 1, 2, and 3, respectively. From these three tables, we have the following observations. First, the three RNGs yield almost the same errors in penetration rate μ and $P(c,k;h)$. Second, MRG32k3a takes almost twice the time as RAND and MT19937. Third, the differences, e.g. $\|P_M(c,k;-1)-\mu\|_1$ and $\|P_M(c,k;h)-P(c,k;h)\|_1$, decrease with increasing number of samples in a rate proportional to $1/\sqrt{10}$. This convergence pattern, observed for all three norms of errors, agrees to that predicted by the fundamental theorem of Monte Carlo simulation (Kalos and Whitlock, 1986, Chapter 4). Since the performance measurements, $S(c,k;h)$ and $s(c,k)$ are derived from $P(c,k;h)$, we expect them to share the same convergence pattern as $P(c,k;h)$. Therefore, we can conclude that the Monte Carlo simulation results converge to the analytical results in (Jin and Recker, 2005). In this sense, the analytical model and the Monte Carlo model cross-validate each other.

5 Conclusion

In this paper, we proposed a Monte Carlo simulation model of instantaneous information propagation through inter-vehicle communication in a traffic stream. With three different, well-chosen random

[7]Ref: http://www.mingw.org/

number generators, we confirmed that Monte Carlo simulation results converge to the theoretical values predicted by an analytical model in (Jin and Recker, 2005). That is, the Monte Carlo simulation model and the analytical model cross-validate each other.

The consistency between the simulation and theoretical results implies that both the Monte Carlo simulation model developed in this paper and the analytical model developed in (Jin and Recker, 2005) are valid for computing the lower bound of the success rate for information to travel beyond a point. That is, we have two parallel approaches to studying the performance of an IVC system: one providing analytical results and the other providing simulation results. Further, although this study is not intended to compare RNGs, it seems that the MT19937 RNG is the best among the three RNGs for Monte Carlo simulations since it is fast with attractive properties.

In the future, we will be interested in studying the success rate of information propagation in a road network and other performance measurements of an IVC system such as its communication capacity (Gupta and Kumar, 2000). For these studies, both the analytical approach in (Jin and Recker, 2005) and the Monte Carlo simulation approach developed in this paper will be exploited.

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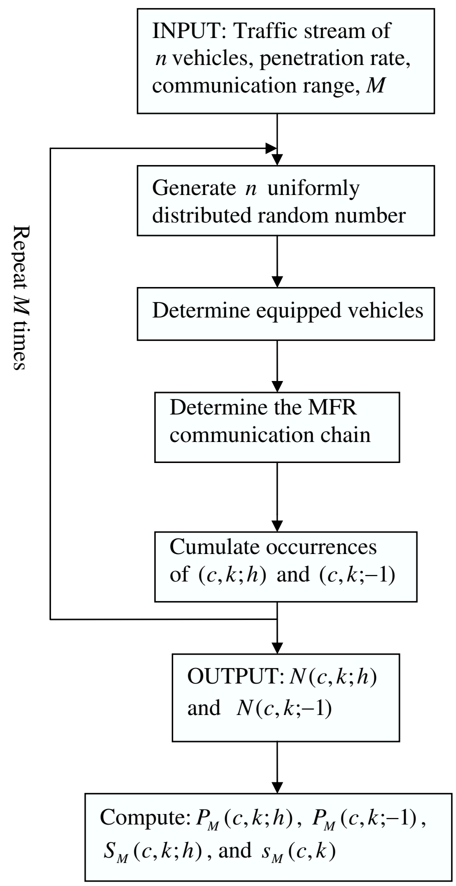

Figure 1: Monte Carlo simulation of instantaneous information propagation through inter-vehicle communications in a traffic stream

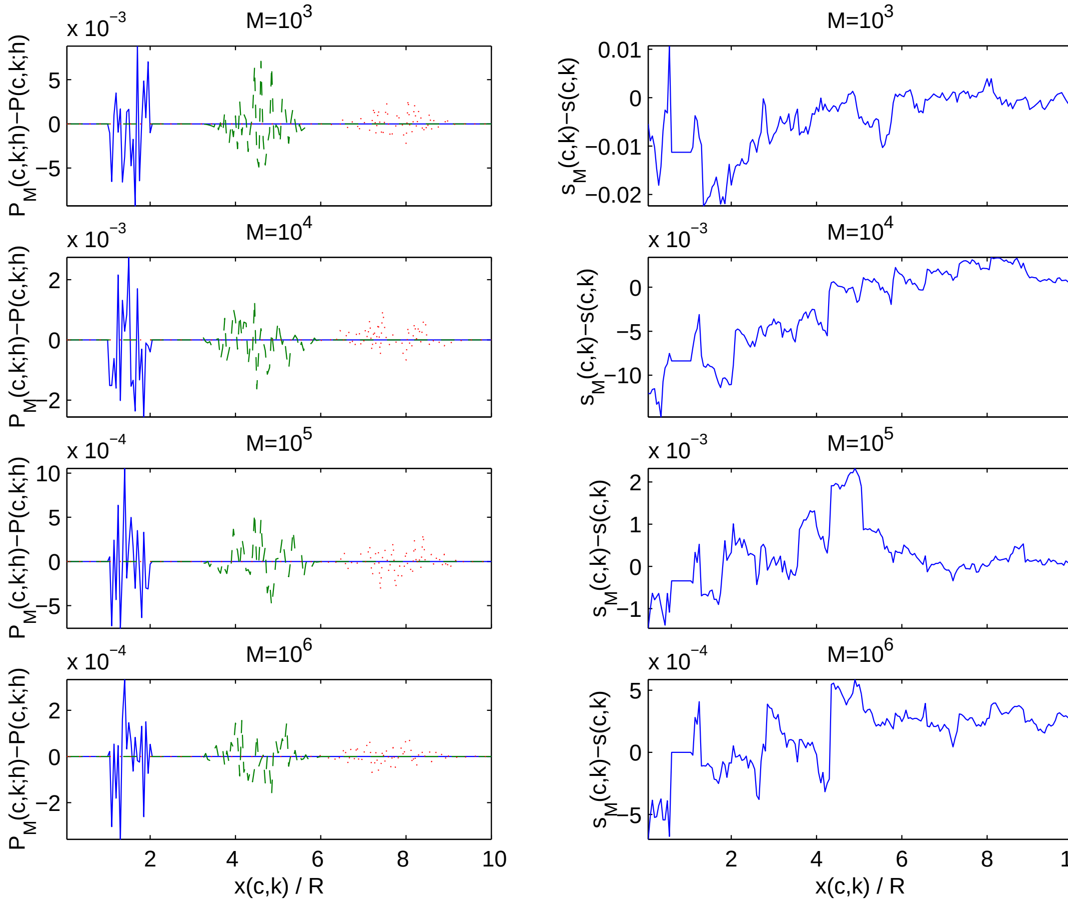

Figure 2: Comparison between Monte Carlo simulation results and theoretical results with MT19937 RNG. (In the left figures, solid, dashed, and dotted lines are for $h = 2$, 6, and 10 respectively.)

M	1e3	1e4	1e5	1e6	1e7	1e8
$\lVert P_M(c,k;-1)-\mu\rVert_1$	1.8323e-4	5.8080e-5	1.8177e-5	5.6776e-6	1.8791e-6	7.0910e-7
$\lVert\cdot\rVert_2$	6.9097e-3	2.1902e-3	6.8543e-4	2.1410e-4	7.0861e-5	2.6740e-5
$\lVert\cdot\rVert_\infty$	2.2000e-2	8.1000e-3	2.3000e-3	7.4300e-4	2.3440e-4	6.6020e-5
$\lVert P_M(c,k;h)-P(c,k;h)\rVert_1$	2.2983e-5	8.4473e-6	2.2481e-6	7.0457e-7	3.0239e-7	7.9274e-8
$\lVert\cdot\rVert_2$	8.6669e-4	3.1854e-4	8.4773e-5	2.6569e-5	1.1403e-5	2.9894e-6
$\lVert\cdot\rVert_\infty$	1.2544e-2	6.0250e-3	1.2713e-3	4.9344e-4	1.6600e-4	5.2160e-5
CPU time (seconds)	1.4100e-1	5.6300e-1	4.8280	5.2438e1	4.8906e2	4.7941e3

Table 1: Monte Carlo simulation results with RAND

M	1e3	1e4	1e5	1e6	1e7	1e8
$\lVert P_M(c,k;-1)-\mu\rVert_1$	1.8205e-4	5.7420e-5	1.8035e-5	5.8041e-6	1.8383e-6	5.7414e-7
$\lVert\cdot\rVert_2$	6.8650e-3	2.1653e-3	6.8007e-4	2.1887e-4	6.9320e-5	2.1650e-5
$\lVert\cdot\rVert_\infty$	2.1000e-2	7.6000e-3	2.4200e-3	7.8500e-4	2.5370e-4	7.4950e-5
$\lVert P_M(c,k;h)-P(c,k;h)\rVert_1$	2.3869e-5	7.6824e-6	2.5533e-6	7.5616e-7	2.7190e-7	7.8789e-8
$\lVert\cdot\rVert_2$	9.0007e-4	2.8970e-4	9.6283e-5	2.8514e-5	1.0253e-5	2.9711e-6
$\lVert\cdot\rVert_\infty$	1.8569e-2	5.6454e-3	1.3900e-3	4.6798e-4	1.9138e-4	4.4170e-5
CPU time (seconds)	2.3500e-1	1.6090	1.5891e1	1.5322e2	1.5328e3	1.5292e4

Table 2: Monte Carlo simulation results with MRG32k3a

M	1e3	1e4	1e5	1e6	1e7	1e8
$\lVert P_M(c,k;-1)-\mu\rVert_1$	1.8412e-4	5.7507e-5	1.8941e-5	5.7828e-6	1.8741e-6	5.8001e-7
$\lVert\cdot\rVert_2$	6.9431e-3	2.1686e-3	7.1425e-4	2.1806e-4	7.0671e-5	2.1872e-5
$\lVert\cdot\rVert_\infty$	2.7000e-2	8.0000e-3	2.3400e-3	1.0190e-3	2.5460e-4	7.8670e-5
$\lVert P_M(c,k;h)-P(c,k;h)\rVert_1$	2.3640e-5	8.0027e-6	2.5177e-6	8.1972e-7	2.2953e-7	7.4876e-8
$\lVert\cdot\rVert_2$	8.9145e-4	3.0178e-4	9.4939e-5	3.0911e-5	8.6553e-6	2.8235e-6
$\lVert\cdot\rVert_\infty$	1.5318e-2	4.5631e-3	1.6600e-3	4.6425e-4	1.2352e-4	5.0290e-5
CPU time (seconds)	1.5700e-1	9.5300e-1	7.7970	7.7594e1	6.9038e2	6.7585e3

Table 3: Monte Carlo simulation results with MT19937